QM1 PhD Qualify

February 11, 2022

1 Problem 1

Consider the Harmonic Oscillator in three-dimensions,

$$H = \sum_{i=1}^{3} \left(\frac{p_i^2}{2m} + \frac{m\omega^2 x_i^2}{2} \right) , \tag{1}$$

use the fact that in position space the momentum operator can be written as $p_i = -i\hbar \frac{\partial}{\partial x_i}$.

- Please derive the ground state, and first excited level wave function (10 points)
- How many states are there in the N-th excited level (10 points), i.e. what is the degeneracy number ?
- Can you explain this degeneracy number from the symmetry of the Hamiltonian ? If so please write out the generators of the symmetry (10 points)

2 Problem 2

Consider the spin-$\frac{1}{2}$ state, with $\vec{S} \cdot \hat{n}|\alpha\rangle = \frac{\hbar}{2}|\alpha\rangle$, where $\hat{n}$ is a unit vector with polar angles (θ, ϕ) as in fig.(2). Using $\vec{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$ where

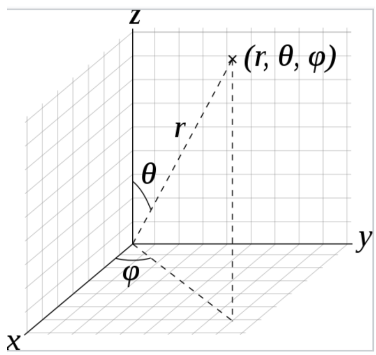
$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} . \tag{2}$$

- Please derive the state $|\alpha\rangle$ (15 points)
- The time reversal operator Ω can be defined as $\Omega\vec{S}\Omega^{-1} = -\vec{S}$. Thus we expect that if

  $$|\alpha'\rangle = \Omega|\alpha\rangle \, , \tag{3}$$

  then

  $$\vec{S} \cdot \hat{n}|\alpha'\rangle = -\frac{\hbar}{2}|\alpha'\rangle \, . \tag{4}$$



Please explain why the above is true (5 points), and use this to derive the form of Ω (10 points). (Hint: you will need to use the anti-linear operator K, which acts as $K(c_a|a\rangle + c_b|b\rangle) = c_a^*|a\rangle + c_b^*|b\rangle$)

3 Problem 3

Consider an electron in a uniform magnetic field in the z-direction. The Hamiltonian is given as

$$H = \frac{1}{2m}\sum_{i=1}^{3}\left(p_i - \frac{eA_i}{c}\right)^2 \equiv \frac{1}{2m}\sum_{i=1}^{3}(\Pi_i)^2 \tag{5}$$

- Please give two distinct choice of (A_1, A_2, A_3), such that one obtains the same magnetic field in z direction (10 points)
- Use this to derive the result of $[\Pi_i, \Pi_j]$ (10 points).
- What is resulting energy eigenvalues ? (20 points)