NTU Examination of PhD Qualifacation (2022)
Classical Mechanics

In the following, $\hat{e}_x,\ \hat{e}_y,\ \hat{e}_z$ are the unit vectors along the $x,\ y,\ z-$axes, respectively.

1. A uniformly distibuted sphere of mass M and radius R is originally rest on the top of an inclined plane with an angle ϕ to the horizontal ground and length l of the inclined side. At time $t=0$ it starts to roll down along the plane without slipping. Let us set the general coordinates x as the distance from the top of the plane to the sphere's contact point to the plane at time t, and θ as the angular displacement between a vertical line through the disk's center and the line connecting its center and the contact point to the plane.

   (a) Set the constraint between x and θ. Introducing a Lagrange undetermined multiplyer λ, write down the sphere's equations of motion and solve $x(t)$ and $\theta(t)$. (Note: the inertia moment of a sphere rotating about any line through its center is $I=\frac{2}{5}MR^2$.) (8%)

   (b) Identify the physical meaning of λ and find the disk's speed when $x=x_0$. (6%)

2. The Hamiltonuan for a system of mass m has the form $H=\frac{p^2q^4}{2m}+\frac{k}{2q^2}$, with k being a constant.

   (a) Find a canonical transformation from $(q,\ p)$ to $(Q,\ P)$ so that the new system will lead to a harmonical oscillator. Verify that it is canonical. (6%)

   (b) A canonical transformation can be reached through several generators. F_2 is the one among them with $p_i=\frac{\partial F_2}{\partial q_i}$; $Q_i=\frac{\partial F_2}{\partial P_i}$.(You don't have to prove them here.) Find the generator $F_2(q,P,t)$ which leads to this transformation. (6%)

   (c) Write the Hamilton's equations of motion for $Q,\ P$ and solve them with the amplitude A to be determined from the total energy E and a phase angle ϕ to be determined from the initial condition. Then find q and p.. (6%)

3. For the system of a certain planet and the Sun with a central force $\vec{f}=-\frac{k}{r^2}\frac{\vec{r}}{r}$, k being a constant, there exists a conserved vector called Laplace-Runge-Lenz vector defined as $\vec{A}_R=\vec{p}\times\vec{L}-mk\frac{\vec{r}}{r}$, with m being the reduced mass of the system, $\vec{r}$ the position vector of the planet to the Sun, and $\vec{p},\ \vec{L}$ the planet's linear and angular momenta around the Sun.

   (a) It can be shown that $\vec{A}_R$ either parallel or anti-parallel to the major axis of the planet's orbit around the Sun(You don't have to prove this.). From A_R, show that the planet's orbit equation is in the form of $\frac{1}{r}=\frac{1}{r_0}(1+\varepsilon cos\theta)$. Your final results should have r_0 and ε written in terms of the conserved quantities $L,\ A_R$, and constants $m,\ k$. (7%)

   (b) Show that $A_R^2=2mL^2E+m^2k^2$, with E being the total (mechanical) energy of the system. From part (a), discuss how the eccentrcity ε varies with E, that is, as $\varepsilon>1,\ =1,\ <1,\ =0$, how E varies. (7%)

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4. A point particle of mass m is connected to three identical massless springs of force constant k, and the other ends of these springs are fixed at points $(1, 1)$, $(-1, 1)$ and $(-1, -1)$ in the $xy-$plane. The particle is originally at its equilibrium point $(0, 0)$. At time $t = 0$, it is pulled slightly to an adjacent point (x, y) and then released, with $x << 1,\ y << 1$.

   (a) Write the original full Lagrangian and then rewrite it appropriate for small oscillations. (5%)

   (b) Find normal mode frequencies and normal coordinates for the particle's motion. (9%)

5. (a) A rigid body contains N particles, with the i^{th} particle of mass m_i and position vector $\vec{r}_i = x_i\hat{e}_x + y_i\hat{e}_y + z_i\hat{e}_z$ $(i = 1, ..., N)$. Its angular momentum is $\vec{L}$ when rotates about a certain axis at the angular velocity $\vec{\omega} = \omega_x\hat{e}_x + \omega_y\hat{e}_y + \omega_z\hat{e}_z$. Write down each component of the inertia tensor in terms of $m_i,\ x_i,\ y_i,\ z_i$. (6%)

   (b) A certain rigid body may be replaced by the three point masses, mass $m_1 = m_0$, at $(1, 1, -2)$, $m_2 = 2m_0$ at $(-1, -1, 0)$, and $m_3 = m_0$ at $(1, 1, 2)$, here coordinates in the unit of r_0= constant, that is, "1" means "r_0", "2" means "$2r_0$", and so on. Find the principal moments of inertia about the origin and a set of principal axes. (8%)

6. A particle of mass m moves in two dimensional space subject to the potential $V(x, y) = k(x^2 + xy + y^2)$, with k being a positive constant.

   (a) Find a canonical transformation to decouple the potential so that the Hamilton's characteristic function W is completely separable. Verify your transformation is indeed canonical, and the write down the Hamilton's characteristic function W after the transformation. (7%)

   (b) Apply the action variables to find the frequencies of the motion. (7%)

7. (a) From relativity, show that in a free system, $(Et - \vec{p}\cdot\vec{r})$ is an invariant in any inertia reference frame, E, $\vec{p}$ being the system's energy and linear momentum, and t the time, $\vec{r}$ being its position. (6%)

   (b) Keeping proper terms for classical limit, show that the time-dependent Schrödinger equation can lead to the Hamilton-Jacobi equation. (6%)