NTU Physics -- Statistical Physics Qualifying Exam (2022)

Please note: 4 problems on 2 pages. Answers in Chinese or English are acceptable.

1. **20pts.** Consider the canonical ensemble.

(a) *10pts.* Show that the energy fluctuation of the second and third order is given by

$$\overline{(E-\bar{E})^2} = k_B T^2 C_V$$

and

$$\overline{(E-\bar{E})^3} = k_B^2 \left(T^4 \left(\frac{\partial C_V}{\partial T}\right)_V + 2T^3 C_V\right)$$

respectively, where E and $\bar{E}$ denote the energy and its mean value, respectively, and C_V is the constant-volume specific heat.

(b) *10pts.* For a monatomic ideal gas, show that

$$\frac{\overline{(E-\bar{E})^2}}{\bar{E}^2} = \frac{2}{3N}$$

and

$$\frac{\overline{(E-\bar{E})^3}}{\bar{E}^3} = \frac{8}{9N^2}.$$

2. **15pts.** Consider a gas of N neutral particles of spin 1/2 and magnetic moment μ, therefore obeying Fermi-Dirac statistics, confined in a volume V. A massive magnetic moment is produced due to alignment of the spins when a magnetic field H is applied.

(a) *5pts.* Find an expression for the magnetic moment per unit volume.

(b) *10pts.* Determine the magnetic susceptibility in the limit of zero magnetic field up to the second order T^2. You may need the integral:

$$\int_0^\infty \frac{\sqrt{x}dx}{e^{(x-a)/(k_B T)}+1} = \frac{2}{3}a^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{k_B T}{a}\right)^2 + \mathcal{O}(T^3)\right].$$

3. **35pts.** Consider a two-dimensional ideal gas of N non-relativistic spinless fermions of mass m in a box of area A.

(a) *5pts.* Determine the density of states $g(\epsilon)$ at energy ϵ.

(b) *10pts.* Write the equation that determines the chemical potential μ of the gas for a given temperature T. In the limit of the fugacity $z = e^{\beta\mu} \ll 1$, where μ is very negative so that $e^{\beta(\epsilon-\mu)} \gg 1$ for any $\epsilon \geq 0$, show that the solution leads to

$$z \approx \frac{N}{N_c}$$

where $N_c = \frac{mk_B TA}{2\pi\hbar^2}$.

(c) *10pts.* Suppose there are also N low-lying degenerate bound states having the same energy $-\epsilon_0$, where $\epsilon_0 > 0$. At temperature T, determine the average numbers of bound particles and the free ones, N_b and N_f, respectively, in terms of the chemical potential μ. Find an expression for N_f in the limit $z \ll 1$.

(d) *10pts.* Write the equation that determines μ. Solve the equation in the limit $z \ll 1$ to determine z and N_f for given T and N.

4. **30pts.** Consider the entropy from the canonical-ensemble point of view.

(a) *10pts.* Show that the entropy $S = k_B(\ln Z_N(T,V) + \beta U)$, where Z_N is the canonical partition function of number N and U is the internal energy.

(b) *10pts.* Further, show that the entropy can be written as $S = -k_B \sum_r p_r \ln p_r$, where $p_r = e^{-\beta\epsilon_r}/Z_N$ with ϵ_r is the corresponding energy of a state r.

(c) *10pts.* Show that, when following the definition of S in (b), the form $p_r \propto e^{-\beta\epsilon_r}$ maximizes S.