Quantum Mechanics (I) 2016 Qualify Exam

1. The Hamiltonian of a point particle in 1-dimensional space is $H = \frac{a^2}{2}\hat{x}^2 + \frac{b^2}{2}\hat{p}^2$, where $\hat{x}$, $\hat{p}$ are the position and momentum operators, and a, b are constant positive numbers.

   (a) (10%) Find the energy spectrum for this quantum mechanical system.

   (b) (10%) Define the momentum eigenstate $|p\rangle$ by $\hat{p}|p\rangle = p|p\rangle$ for some real constant p with the normalization condition $\langle p|p'\rangle = \delta(p-p')$. Find $\langle x|p\rangle$, where $|x\rangle$ is the eigenstate of $\hat{x}$ normalized by $\langle x|x'\rangle = \delta(x-x')$.

   (c) (10%) Find the ground state wave function $\psi_0(x,t)$ in the representation $\hat{x} = x$, $\hat{p} = -i\hbar\frac{d}{dx}$, in the Schrödinger picture. Remember to normalize the wave function.

2. The Hilbert space of a quantum system is 2-dimensional and the Hamiltonian is $H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ for real constants a, b.

   (a) (10%) In the Schrödinger picture, find the state $\psi(t)$ for general t if $\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   (b) (5%) In the Heisenberg picture, what is the operator $A(t)$ if $A(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

   (c) (5%) If this quantum system and another quantum system with the Hamiltonian $H = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ are put together as a composite system, what is the spectrum of the total Hamiltonian? (Assume that the interaction between the two sub-systems can be ignored.)

3. (20%) Let operators A, B satisfy the commutation relation $[A, B] = 1$.

   (a) Simplify $e^A e^{B^2} e^{-A}$.

   (b) Let $C = AB$. For given positive integers n and m, find the function $F(C)$ in the identify $A^n C^m = F(C)A^n$.

4. (20%) In the WKB approximation of a point particle in 1-dimensional space, for a given Hamiltonian $H = \frac{p^2}{2m} + V(x)$, we write the wave function as $\psi(x,t) = e^{iS/\hbar}e^{-iEt/\hbar}$, with energy E and $S = W_0(x) + \hbar W_1(x) + \hbar^2 W_2(x) + \cdots$. Use the Schrödinger equation to determine W_0 and W_1. Write the approximate wave function in the form $\psi \simeq Ae^{iW_0/\hbar}$, where $A = e^{iW_1}$, for the region $E > V(x)$, ignoring terms of order $\hbar^2$ or higher in S.

5. (10%) Define $|\hat{\mathbf{n}}; +\rangle$ as the *normalized* eigenstate of the operator $\hat{\mathbf{n}} \cdot \mathbf{s}$ with the eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector on the x-z plane that makes an angle θ with the positive z-axis. Express this state $|\hat{\mathbf{n}}; +\rangle$ as a superposition of the spin-up state $|\uparrow\rangle$ and spin-down state $|\downarrow\rangle$, which are eigenstates of s_z.