Qualification Exam

1. (25 points) The Lagrangian for a relativistic free particle of mass m in $3+1$-dimensional Minkowski spacetime is

$$L = -mc^2\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}}; \qquad \mathbf{v} \equiv \frac{d\mathbf{x}}{dt} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \tag{1}$$

where c is the speed of light.

(1) Derive the equation of motion and find the general solution to it.

(2) Express the momentum $\mathbf{p}$ conjugate to $\mathbf{x}$ as a function of $m, c, \mathbf{x}, \mathbf{v}$.

(3) Express the Energy E as a function of $m, c, \mathbf{x}, \mathbf{v}$. Show that when $|\mathbf{v}| \ll c$ the energy E is approximately the sum of the rest mass energy and the (non-relativistic) kinetic energy.

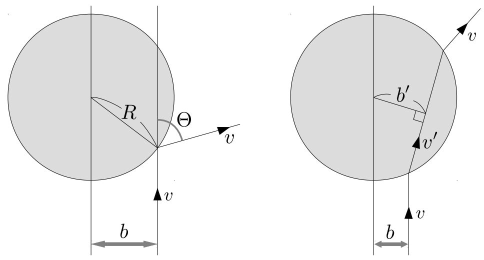
(4) The Lagrangian for a particle with charge q under a constant electric field $\mathbf{E}$ is

$$L = -mc^2\sqrt{1-\frac{|\mathbf{v}|^2}{c^2}} + q\mathbf{E}\cdot\mathbf{x}. \tag{2}$$

For $\mathbf{E} = (a, 0, 0)$, solve the equation of motion for the particle moving in the x-direction.

2. (25 points) Consider the scattering of a point particle of mass m by a central force potential

$$V(r) = \begin{cases} 0 & (r > R) \\ E & (r \le R) \end{cases} \tag{3}$$



which takes place in a two-dimensional plane.

Assume the potential is repulsive ($E > 0$). The incident particle has velocity v and is along a line at a distance b from the center of the force.

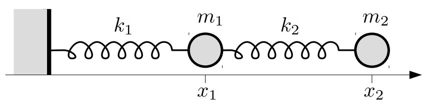
(1) What are the kinetic energy and the angular momentum (about the center of the force) of the incident particle?

(2) For $E = \infty$ the particle cannot go inside the disk of nonzero potential, as shown in the left of the figure. Express the scattering angle Θ using R and b.

(3) Suppose the kinetic energy of the particle is greater than E and the particle goes inside the disk as shown in the right of the figure. Find b' and v'.

(4) Under the same assumption as (3), find the value of b at which the scattering angle Θ is the largest, and find the maximum value of Θ.

3. (25 points) Consider a particle of mass m_1 connected to a wall by a spring of force constant k_1, and another particle of mass m_2 connected to it by a spring of force constant k_2.



We consider the one-dimensional motion of the two particles. The positions of the two particles are parameterized by x_1, x_2 ($x_1 = x_2 = 0$ at equilibrium).

(1) Write down the Lagrangian and derive the equation of motion.

(2) The solution to the equation of motion should take the form

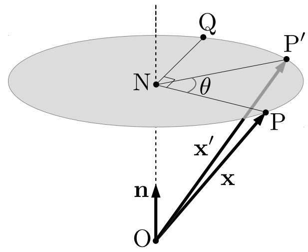
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t + \delta) \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos(\tilde{\omega} t + \tilde{\delta}) \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{bmatrix}, \tag{4}$$

where $a_i, \tilde{a}_i, \omega, \tilde{\omega}, \delta, \tilde{\delta}$ are constants. Derive the equation (a_i, ω) have to satisfy. (The other set $(\tilde{a}_i, \tilde{\omega})$ has to satisfy the same equation).

(3) Find the general solution for the case

$$(m_1,\, m_2,\, k_1,\, k_2) = (2m,\, m,\, 2m\Omega^2,\, m\Omega^2) \tag{5}$$

4. (25 points) We wish to construct a 3×3 orthogonal matrix $\mathbf{A}$ that rotates any vector about an axis along a unit vector $\mathbf{n}$ ($|\mathbf{n}| = 1$) by an angle θ. The figure shows that



$$\mathbf{A}\mathbf{x} = \mathbf{x}', \tag{6}$$

where $\mathbf{x} = \overrightarrow{\mathrm{OP}}$ and $\mathbf{x}' = \overrightarrow{\mathrm{OP}'}$. The matrix $\mathbf{A}$ for $\theta = \pi$ transforms $\overrightarrow{\mathrm{OP}}$ to $\overrightarrow{\mathrm{OQ}}$. The plane which contains $\mathrm{P}, \mathrm{P}', \mathrm{Q}$ intersects orthogonally with the rotation axis at N.

(1) Express $\overrightarrow{\mathrm{ON}}$, $\overrightarrow{\mathrm{NP}}$ and $\overrightarrow{\mathrm{NQ}}$ using $\mathbf{x}$ and $\mathbf{n}$.

(2) Express $\mathbf{x}' = \overrightarrow{\mathrm{OP}'}$ as a linear combination of the three vectors $\mathbf{x}$, $\mathbf{n}$ and $\mathbf{n} \times \mathbf{x}$.

(3) Find out the 3×3 matrix $\mathbf{A}$ and write its components using $\mathbf{n} = (n_1, n_2, n_3)$ and θ.

(4) $\mathbf{A}$ can be written as $\mathbf{A} = \exp(\boldsymbol{\Theta})$ for a certain 3×3 matrix $\boldsymbol{\Theta}$. Find $\boldsymbol{\Theta}$.