Classical Electrodynamics I

* *All quantities are expressed in terms of SI units.*
* *Vectors are represented with right-pointing arrow notation above their names, as in $\vec{v}$.*

1. [10%] In electrostatics, Gauss's law $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}$ is equivalent to
$$\vec{E}(\vec{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}')(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}d^3x'.$$
In a dielectric material, the free charge density is related to $\vec{D}$ by $\vec{\nabla}\cdot\vec{D}=\rho_f$. Can we also conclude that
$$\vec{D}(\vec{x})=\frac{1}{4\pi}\int\frac{\rho_f(\vec{x}')(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}d^3x'?$$
Why? (A yes or no answer will not earn you any credit. You must provide an explanation.)

2. [10%] Why is Ampère's Law, $\vec{\nabla}\times\vec{H}=\vec{J}$, no longer valid when charge density is time dependent?

3. (a) [10%] What are the four macroscopic Maxwell equations (The equations that relate macroscopic fields $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$ and sources ρ_f, $\vec{J}_f$)?
   (b) [10%] Up to the dipole approximation, express $\vec{D}$ in terms of $\vec{E}$ and the polarization $\vec{P}$, and express $\vec{H}$ in terms of $\vec{B}$ and the magnetization $\vec{M}$.

4. [20%] Find the electrostatic potential outside a charged metal sphere (charge Q, radius R) placed in an otherwise uniform electric field $\vec{E}_0$. Explain clearly where you are setting the zero of potential.

5. The vector potential due to a static magnetic dipole $\vec{m}$ located at the origin is $\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{r^3}=-\frac{\mu_0}{4\pi}\vec{m}\times\vec{\nabla}\left(\frac{1}{r}\right)$ where $r=|\vec{x}|$.
   (a) [5%] If $r\neq 0$, show that $\vec{B}=\vec{\nabla}\times\vec{A}=\frac{\mu_0}{4\pi}\frac{3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}}{r^3}$ where $\hat{n}=\frac{\vec{x}}{r}$.
   (b) [5%] For $r\neq 0$, show that vector potential $\vec{A}$ satisfies the Laplace equation, $\nabla^2\vec{A}=0$, and we may write $\vec{A}(\vec{x})=\sum_{m=-1}^{1}\frac{\vec{A}_{1m}}{r^2}Y_{1m}(\theta,\phi)$ as a linear combination of spherical harmonics Y_{1m}.
   (c) [5%] For $r\neq 0$, show that the magnetic field $\vec{B}\equiv\vec{\nabla}\times\vec{A}$ also satisfies $\nabla^2\vec{B}=0$, and $\vec{B}$ can be written as $\vec{B}(\vec{x})=\sum_{m=-2}^{2}\frac{\vec{B}_{2m}}{r^3}Y_{2m}(\theta,\phi)$, a linear combination of spherical harmonics of order 2.
   (d) [5%] Prove that the surface integral $\oint_S\vec{B}d^2a=0$, where S is the spherical surface of radius R centered at the origin.

6. The electric field $\vec{E}$ may be decomposed as $\vec{E}=\vec{E}_s+\vec{E}_{induced}$ where $\vec{\nabla}\times\vec{E}_s=0$ and $\vec{\nabla}\cdot\vec{E}_{induced}=0$. Kirchhoff's loop rule states that the directed sum of the electrical potential differences (voltage) ΔV around any closed network is zero. But for a circuit with an inductor, $\vec{\nabla}\times\vec{E}\neq 0$ and $\vec{E}\neq-\vec{\nabla}V$. Yet we can still find a potential V so that $\vec{E}_s=-\vec{\nabla}V$ and continue to treat the circuit containing inductors with Kirchhoff's loop rule.
   (a) [10%] Show that $\vec{E}_{induced}=-\frac{\partial\vec{A}}{\partial t}$ and $\vec{A}$ is the vector potential in the Coulomb gauge.
   (b) [5%] By definition, the voltage change across an inductor is the negative of the line integral of $\vec{E}_s$ across the inductor: $\Delta V_L=-\int_{inductor}\vec{E}_s\cdot d\vec{s}$. If the current flows through the inductor via perfect conducting wire (with infinite conductivity), show that $\Delta V_L=\int_{inductor}\vec{E}_{induced}\cdot d\vec{s}$.
   (c) [5%] For a perfect conducting inductor with no leakage of magnetic flux ($\vec{B}=0$ outside the inductor), and with negligible $\vec{E}_{induced}$ outside the inductor so that we may extend $\int_{inductor}\vec{E}_{induced}\cdot d\vec{s}$ to a closed-loop line integral $\oint_{loop}\vec{E}_{induced}\cdot d\vec{s}$, show that $\Delta V_L=-L\frac{di}{dt}$, where L is the inductance.